Neutron Star Core Equations of State and the Maximum Neutron Star Mass

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Introduction

Neutron stars are the compact remnants of massive stars after they undergo core collapse. As the core collapses and density increases, protons are converted into neutrons via beta capture. If the progenitor star is very massive, the core cannot stop collapsing before reaching its Schwarszchild radius and a black hole is formed. However, for a range of progenitor star masses, the core is supported against collapse by neutron degeneracy pressure at a radius of $\sim 10\ km$. The gravitational energy released by the core's collapse to such a small radius powers a core collapse supernova.

The equation of state of neutron star cores remains a great mystery. The interactions between nucleons are governed by the strong nuclear force. Quantum chromodynamics (QCD) governs the strong interactions between the quarks and gluons that comprise nucleons, but is intractable for calculations at low temperature and high density. Neutron star cores contain matter denser than atomic nuclei ($\rho_0 = 2.8 \times 10^{14}\ g\ cm^{-3}$), thus terrestrial experiments on nuclei yield limited insight to the equations of state of neutron star cores.

An effective nuclear field theory is used to calculate these equations of state. The interactions are only directly testable at nuclear densities, leading to uncertainties at higher densities. The many body problem cannot be solved exactly, necessitating approximations. The methods of calculation can be classified into two branches. In the first, interactions are calculated between particles in vacuum using a perturbation expansion in Feynman diagrams. In the second, interactions in a dense baryonic medium are rewritten into an effective Hamiltonian.

Neutron stars are so compact that the stellar structure equations must also be modified to account for General Relativity. Solving the Einstein equations for a time-invariant, spherically symmetric for a fluid in hydrostatic equilibrium yields the Tolman-Oppenheimer-Volkoff equation. Effects of General Relativity also establish a maximum stable neutron star mass which depends strongly on the "stiffness" of the equation of state. Equations of state with higher pressures for a given energy density are said to be "stiffer" (and the opposite deemed "softer"); stiffer equations of state produce higher maximum masses.

Neutron star cores may contain exotic components beyond nucleons and leptons at densities beyond the nuclear density. Nucleons may convert into hyperons (baryons with strange quarks) via strangeness changing weak interactions $u + e \rightarrow s + \nu_e$. Pions and kaons, mesons of nuclear field theory, could also form condensates. Finally, quarks may become deconfined from baryons.

This work will focus on the constraints placed by the maximum neutron star mass. First, nucleonic compositions will be discussed and stellar models constructed for the Heiselberg & Hjorth-Jensen equations of state. Then, exotic compositions will be qualitatively discussed. The measured mass of millisecond pulsar J1614-2230 of $1.97 \pm 0.04\ M_{Sol}$ (Demorest 2010) strongly constrains exotic equations of state; some recent literature concerning these constraints will be highlighted.

Nucleonic Equations of State

Perhaps the simplest nucleonic equations of state are those which exclude the nucleonic interactions. After deriving the Tolman-Oppenheimer-Volkoff equation Oppenheimer and Volkoff (1939) created neutron star models assuming an ideal neutron gas, yielding a maximum mass of $0.71\ M_{Sol}$. However, this mass is smaller than the stellar cores that collapse into neutron stars. Harrison et al. (1958) considered a neutron, proton, electron ideal gas in equilibrium with the beta decay and capture processes. The inclusion these components slightly softened the equation of state, yielding a maximum mass of $0.70\ M_{Sol}$. Cameron (1959) pointed out, using effective nucleon interactions by Skyrme (1959), that nuclear interactions could stiffen the equation of state. Repulsive interactions would increase pressure and allow higher maximum masses of about $2.0\ M_{Sol}$.

Nucleonic equations of state can be tested at the density ρ_0 through nuclear experiments. Since the strong nuclear force has isospin symmetry, if protons and neutrons are switched, the strong interaction should remain unchanged. Thus, the strong interaction should only involve even powers of the neutron fraction excess $x_n - x_p = 1 - 2x_n$. Since experiments test nuclear interactions with nuclei at around ρ_0 and almost even proton and neutron numbers, the energy from nucleonic interactions is commonly written in a Taylor series in ρ around ρ_0 and $1 - 2x_p$ around 0 (Lattimer 2012)

$$\epsilon_N(n_n, n_p) = -B + \frac{K_0}{18}(u-1)^2 + \frac{K_0'}{162}(u-1)^3 + S_2(u)\left(1 - 2x_p\right)^2$$

Where $u = \frac{n_b}{n_0} = \frac{n_n + n_p}{n_0}$ with n_0 and $x_i = \frac{n_i}{n_b}$. The binding energy B, incompressibility K_0, skewness $K_0{}'$ are measureable. $S_2(u)$ is the $\mathcal{O}([1-2x]^2)$ term of the symmetry energy $S(u)$, which is also measurable.

We consider a class of equations of state (including arguments from Haensel et al. 2007) with an analytic expression for the interaction energy in the similar form

$$E_N = W(n_b) + \frac{S(n_b)}{n_b^2}\left(n_n - n_p\right)^2$$

With $E = \frac{\epsilon}{n_b}$, the energy per nucleon.

The electrons and muons are treated as ideal non-interacting Fermi gases. Thus, the total energy per nucleon also consists of the rest mass energy of the nucleons (E_{N0}), energy of the electrons (E_e), and energy of the muons (E_μ)

$$E = E_N(n_n, n_p) + E_{N0}(n_n, n_p) + E_e(n_e) + E_\mu(n_\mu)$$

Beta equilibrium relates the chemical potentials of all the components. Assuming that the neutron star is transparent to neutrinos, $\mu_{\nu_e} = \mu_{\nu_\mu} = 0$, and thus

$$\begin{array}{lll} p + e \to n + \nu_e & n + \bar{e} \to p + \overline{\nu_e} & \Rightarrow \mu_n = \mu_p + \mu_e \\ p + \mu \to n + \nu_\mu & n + \bar{\mu} \to p + \overline{\nu_\mu} & \Rightarrow \mu_n = \mu_p + \mu_\mu \end{array} \Rightarrow \mu_e = \mu_\mu$$

At neutron star densities the electrons are ultra-relativistic, whereas the muons are somewhat relativistic (if they are present at all)

$$\mu_e = cp_{F,e} \quad \mu_\mu = \sqrt{m_\mu^2 c^4 + p_{F,\mu}^2 c^2} \quad p_{F,\ell} = \hbar(3\pi^2 n_b x_\ell)^{\frac{1}{3}}$$

$$\mu_e = \mu_\mu \Rightarrow x_e^{\frac{2}{3}} - x_\mu^{\frac{2}{3}} - \left(\frac{m_\mu c^2}{\hbar}\right)^2 (3\pi^2 n_b)^{\frac{-2}{3}} = 0$$

Using the condition of beta equilibrium $\mu_n = \mu_p + \mu_e$

$$n_b \left(\frac{\partial E}{\partial n_n}\right)_{n_p} = n_b \left(\frac{\partial E}{\partial n_p}\right)_{n_n} + \mu_e$$

Using $n_b = n_n + n_p$ in taking the derivatives

$$\frac{\partial W}{\partial n_b} + \frac{\partial}{\partial n_b}\left(\frac{S(n_b)}{n_b^2}\right)(n_n - n_p)^2 + 2\frac{S(n_b)}{n_b^2}(n_n - n_p) + m_p c^2$$
$$= \frac{\partial W}{\partial n_b} + \frac{\partial}{\partial n_b}\left(\frac{S(n_b)}{n_b^2}\right)(n_n - n_p)^2 - 2\frac{S(n_b)}{n_b^2}(n_n - n_p) + m_n c^2 + \frac{\mu_e}{n_b}$$

Cancelling similar terms on both sides and taking $m_p \approx m_n$

$$\mu_e = 4\frac{S(n_b)}{n_b^2}(n_n - n_p) \Rightarrow \frac{\mu_e}{4S(n_b)} = 1 - 2x_p \Rightarrow x_p = \frac{1}{2} - \frac{\mu_e}{8S(n_b)}$$

Now using the electric neutrality condition $x_p = x_e + x_\mu$

$$\frac{1}{2} - \frac{\mu_e}{8S(n_b)} = x_e + x_\mu \Rightarrow x_e + x_\mu - \frac{1}{2} + \frac{\hbar c(3\pi^2 n_b x_e)^{\frac{1}{3}}}{8S(n_b)} = 0$$

So we have two equations to determine x_e, x_μ for each nucleonic density n_b. Using the nuclear saturation density $n_0 = 0.16\, fm^{-3}$ gives

$$x_e^{\frac{2}{3}} - x_\mu^{\frac{2}{3}} - \left(\frac{m_\mu c^2}{\hbar}\right)^2 (3\pi^2 n_b)^{\frac{-2}{3}} = 0 \qquad \Rightarrow x_e^{\frac{2}{3}} - x_\mu^{\frac{2}{3}} - 0.1016u^{-\frac{2}{3}} = 0$$

$$x_e + x_\mu - \frac{1}{2} + \frac{\hbar c(3\pi^2 n_b x_e)^{\frac{1}{3}}}{8S(n_b)} = 0 \qquad \Rightarrow x_e + x_\mu - \frac{1}{2} + \frac{41.43\ MeV}{S(u)} u^{\frac{1}{3}} x^{\frac{1}{3}} = 0$$

Muons are only present when $x_\mu > 0 \Rightarrow x_e^{\frac{2}{3}} > 0.1016u^{-\frac{2}{3}}$. Numerically, we can find the u at which $x_e^{\frac{2}{3}} = 0.1016u^{-\frac{2}{3}}$; below this density, muons are absent and x_e is determined solely by

$$x_e - \frac{1}{2} + \frac{41.43\ MeV}{S(u)}\ u^{\frac{1}{3}} x^{\frac{1}{3}} = 0$$

We will consider $W(u), S(u)$ obtained by Heiselberg and Hjorth-Jensen (2000) from fitting to the Akmal et al. (1998) equation of state

$$W(u) = -15.8\ MeV \frac{u\,(2+\delta-u)}{1+\delta u}$$

$$S(u) = 32\ MeV\ u^{0.6}$$

Which corresponds to $B = -15.8\ MeV$. $0.13 \leq \delta \leq 0.31$ determines the stiffness of the equation of state: low δ gives a stiffer equation of state. δ is also related to the incompressibility K_0

$$K_0 = \frac{18|B|}{1+\delta}$$

The composition of the matter at each u is independent of δ. The density at which muons appear is $u = 0.831$ for the Heiselberg and Hjorth-Jensen (hereforth HHJ) equations of state.

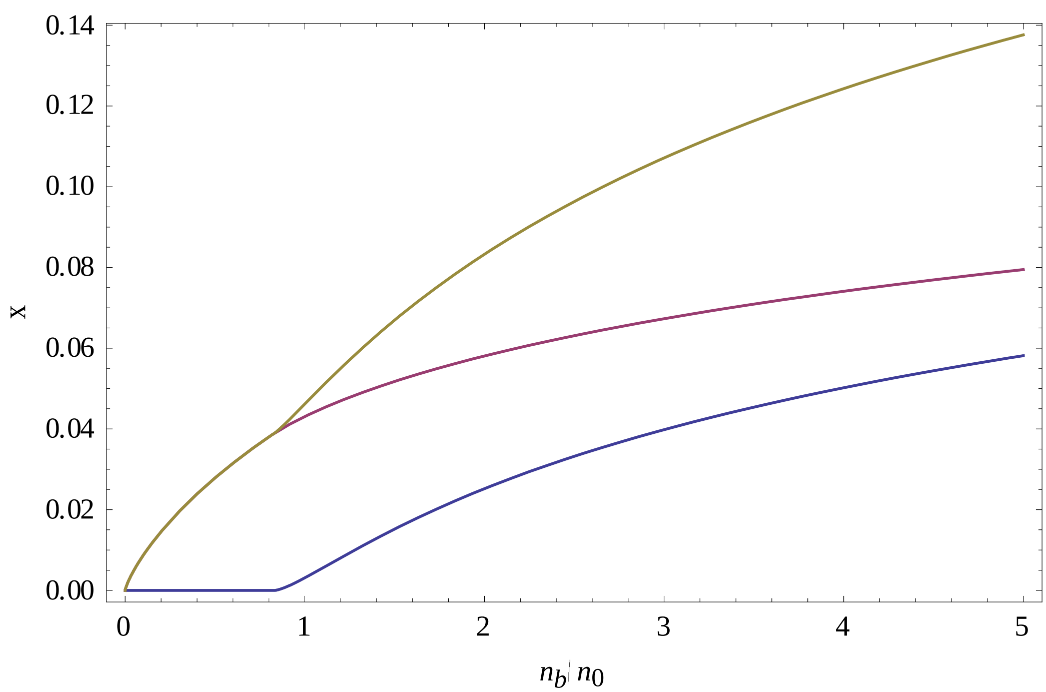

Figure 1. Proton (gold), electron (crimson), and muon (blue) fractions as a function of density in the HHJ equation of state.

Since the muons are an ultrarelativistic Fermi gas and the muons are a Fermi gas

$$\epsilon_\mu = E_\mu\, n_b = \frac{2}{h^3}\int_0^{p_{F,\mu}} 4\pi p^2 \sqrt{m_\mu^2 c^4 + p^2 c^2}\, dp \qquad \epsilon_e = E_e n_b = \frac{2c}{h^3}\int_0^{p_{F,e}} 4\pi p^3\, dp$$

$$P_\mu = \frac{2}{h^3}\int_0^{p_{F,\mu}} 4\pi p^2 \frac{1}{3}\frac{p^2}{\sqrt{m_\mu^2 c^4 + p^2 c^2}}\, dp \qquad P_e = \frac{1}{3}\epsilon_e$$

If p and p_F are measured in MeV, $p_{F,\ell} = 226.3\ MeV\ u^{\frac{1}{3}}\, x_\ell^{\frac{1}{3}}$ and

$$\frac{\epsilon_\mu}{n_0} = 8.24\times 10^{-8} MeV \int_0^{p_F} p^2\sqrt{106^2 + p^2}\, dp \qquad \frac{\epsilon_e}{n_0} = 8.24\times 10^{-8} MeV \int_0^{p_F} p^3\, dp$$

$$\frac{P_\mu}{n_0} = \frac{8.24\times 10^{-8} MeV}{3}\int_0^{p_F} \frac{p^4}{\sqrt{106^2 + p^2}}\, dp \qquad \frac{P_e}{n_0} = \frac{1}{3}\frac{\epsilon_e}{n_0}$$

The nucleonic pressure can be found via the thermodynamic relation

$$P_N = n_b^2\left(\frac{\partial E_N}{\partial n_b}\right)_{eq} = n_b^2\left(\frac{\partial W}{\partial n_b} + \frac{\partial S}{\partial n_b}\left(x_n - x_p\right)_{eq}\right) \Rightarrow \frac{P_N}{n_0} = u^2\left(\frac{\partial W}{\partial u} + \frac{\partial S}{\partial u}\left(x_n - x_p\right)_{eq}\right)$$

So for every density u, we can calculate ϵ and P by adding the nucleonic and leptonic contributions.

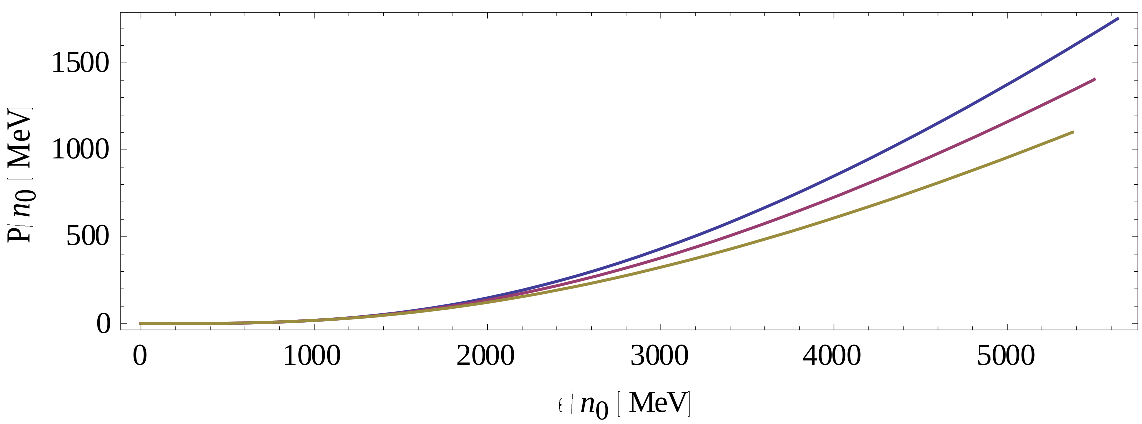

Figure 2. Calculated HHJ equation of state for $\delta = 0.13, 0.2, 0.31$ (blue, crimson, gold).

We can also calculate the speed of sound and confirm that it is subluminal.

$$c_s^2 = \frac{\partial P}{\partial \epsilon}$$

The adiabatic index of the equation of state is around ranges from 2.3 to 3.3.

$$\gamma = \frac{P + \epsilon}{P}\frac{\partial P}{\partial \epsilon}$$

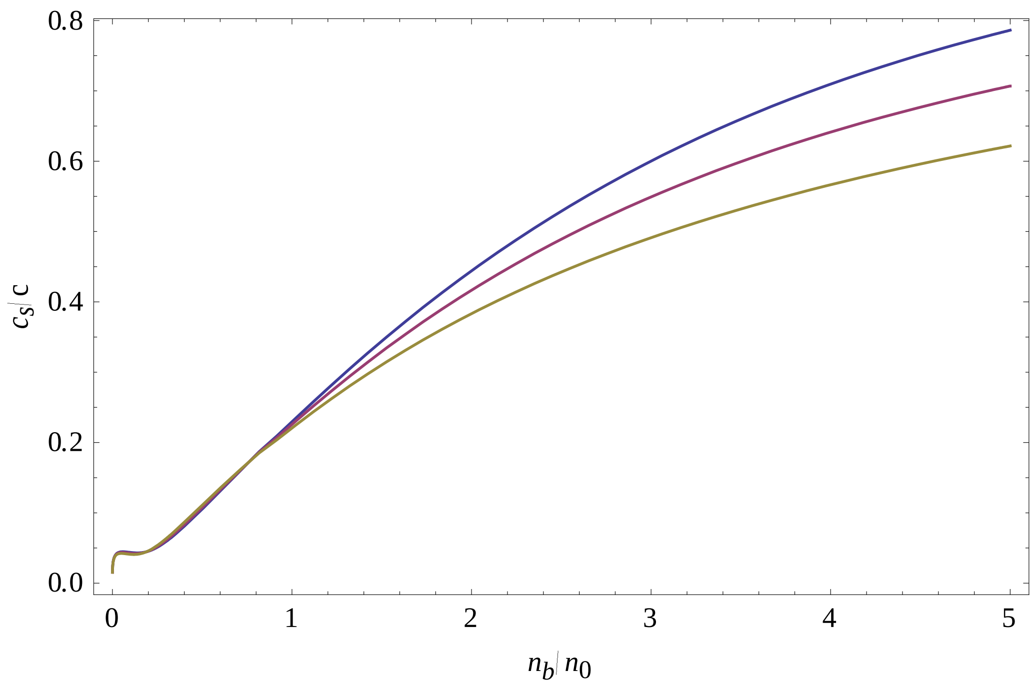

Figure 3. Speed of sound of the HHJ equation of state for $\delta = 0.13, 0.2, 0.31$ (blue, crimson, gold).

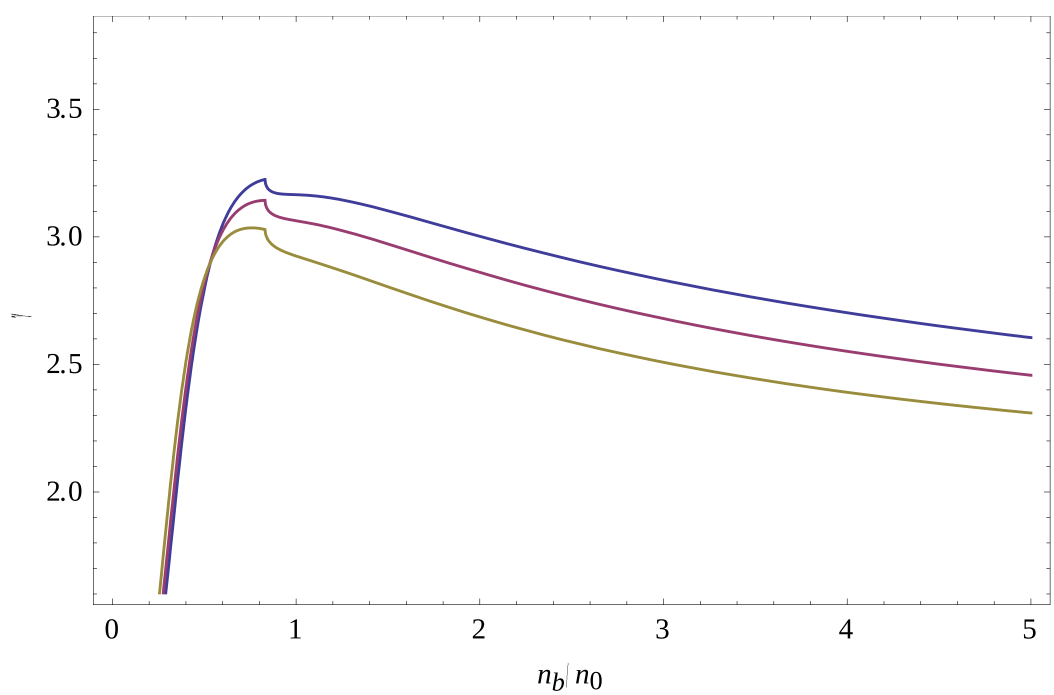

Figure 4. Adiabatic index of the HHJ equation of state for $\delta = 0.13, 0.2, 0.31$ (blue, crimson,gold).

The neutron star equation of state is determined by the Tolman-Oppenheimer-Volkoff equation for hydrostatic equilibrium, mass conservation, and the equation of state

$$\frac{dP}{dr} = -\frac{G}{c^4}\frac{1}{r^2}(\epsilon + P)(M_r c^2 + 4\pi r^3 P)\left(1 - \frac{2GM_r}{cr}\right)^{-1}$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \quad P(\epsilon)$$

We solve the structure equations in the following crude way. A boundary condition on the central pressure P_C of the neutron star is established. Then the approximation is made for small Δr that

$$P_{r=\Delta r} = P_C \quad M_{\Delta r} = \frac{4\pi}{3} r^3 \frac{\epsilon_c}{c^2}$$

Then the structure of the star is constructed in steps of Δr

$$M_{(n+1)\Delta r} = M_{n\Delta r} + 4\pi (n\Delta r)^2 \Delta r \frac{\epsilon_{r=n\Delta \mathrm{r}}}{c^2}$$

$$P_{r=(n+1)\Delta r} = P_{r=n\Delta r} + \left(\frac{dP}{dr}\right)_{r=n\Delta r} \Delta r$$

Continuing until $P < 0$, at which the boundary condition $P_{r=R} = 0$ is deemed to be satisfied. Since neutron stars are $\sim 10\ km$ in radius, Δr of $100\ m$ is chosen to get ~ 100 points of resolution in the model. We create neutron star models for a range of P_C, $500\ MeV\ n_0 < P_C < 5000\ MeV\ n_0$, corresponding to $3.3 < u < 8.6$ with parameter $\delta = 0.13, 0.16, 0.19, 0.22, 0.25, 0.28, 0.31$.

The static stability criterion is a necessary, but not sufficient, condition for the star to be stable against pulsation. Since ϵ_C increases with P_C in the range considered (Haensel et al. 2007)

$$stability \Rightarrow \frac{dM}{d\epsilon_C} > 0 \Leftrightarrow \frac{dM}{dP_C} > 0$$

ie.

$$\frac{dM}{dP_C} < 0 \Leftrightarrow \frac{dM}{d\epsilon_C} < 0 \Rightarrow instability$$

Also, the first mode of pulsation changes stability if and only if $M(R)$ reaches an extremum while $\frac{dR}{d\epsilon_C} < 0$ (Haensel et al. 2007). In the range of P_C considered, R decreases with increasing ϵ_C. For all δ considered, $M(R)$ reaches an extremum at $u \approx 7.8$ (the left most points on Figure 5): the models with smaller radii are unstable since $\frac{dM}{dP_C} < 0$; thus, the models with larger radii must be stable.

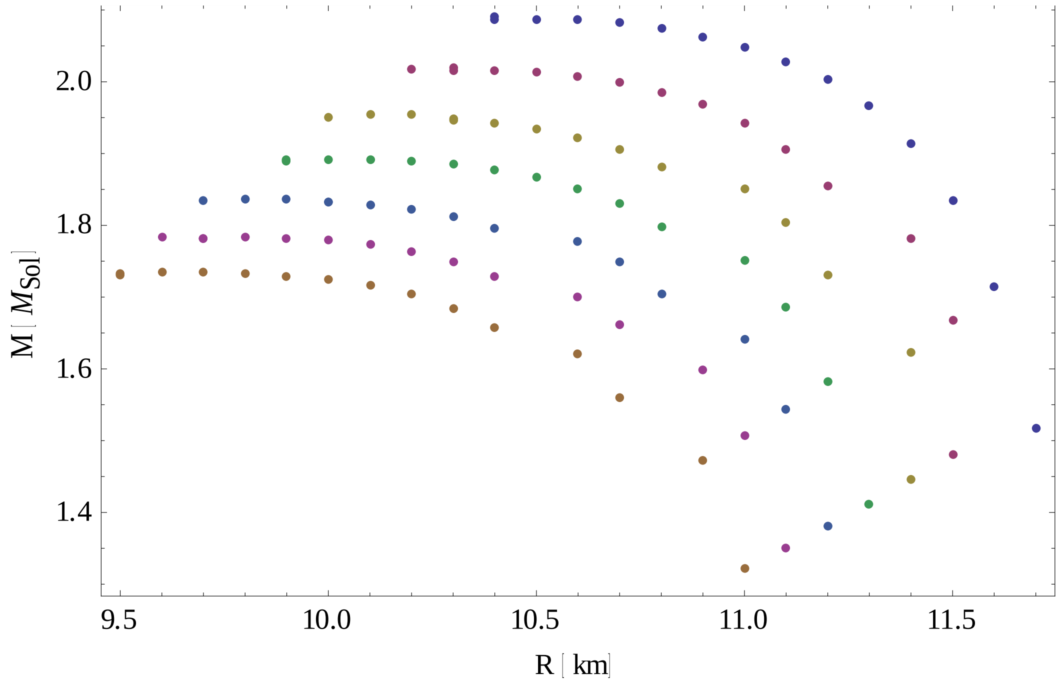

Figure 5. Neutron star $M - R$ relations for $\delta = 0.13, 0.16, 0.19, 0.22, 0.25, 0.28, 0.31$ (top to bottom: blue, crimson, gold, green, light blue, purple, brown). Increasing P_C goes from right to left $500\ MeV\ n_0 < P_C < 4000\ MeV\ n_0$. All models shown are stable, but other stable models with greater radius and lower mass (lower P_C) are not shown.

The mass of J1614-2230 excludes the $\delta \geq 0.29$ models to over 5σ, corresponding to $K_0 > 221\ MeV$.

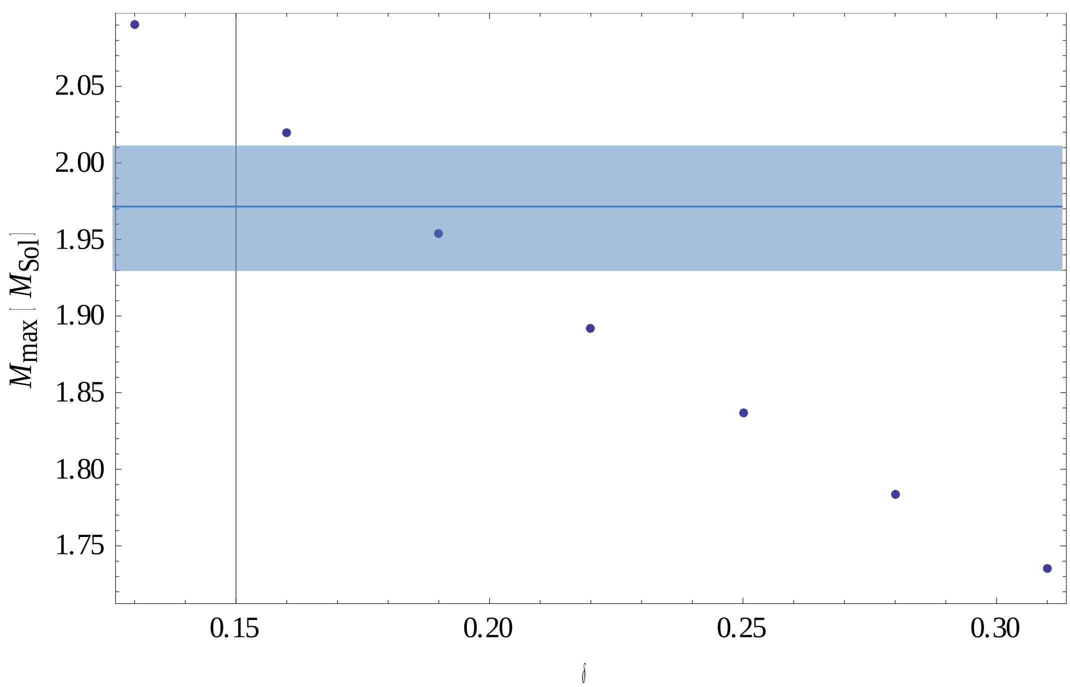

Figure 6. Maximum neutron star mass as a function of δ. The mass of J1614-2230 ($1.97 \pm 0.04\ M_{Sol}$) is shown with a 1σ bar.

Exotic Equations of State

Exotic particles may appear at densities a few times the nuclear density ρ_0. If these exotic particles are allowed to appear by the equation of state, they appear when the chemical potential of the nucleons rises above the energy required to create the exotic particle (rest mass and interaction energy). Equations of state that allow more exotic particles tend to be softer: the nucleon kinetic energy can become exotic particle rest mass, decreasing pressure.

Because hyperonic equations of state are soft, they are constrained by the mass of J1614-2230. Ellis et al. (1990) considered equations of state including hyperons and allowing for the presence of ϕ mesons, as well as first and second order transitions to deconfined quarks. None of their exotic equations of state can reach $1.97M_{Sol}$. Weissenborn et al. (2012a) note that this mass may be reached if a repulsive vector meson, motivated by proposing a SU(3) flavour symmetry (2012b), is introduced, stiffening the equation of state. Katayama et al. (2012) independently find the same result for SU(3) flavour symmetry. Zdunik and Haensel (2012) find that a neutron star with a colour superconducting quark core could reach this mass.

Conclusion

Determining the neutron star equation of state is fraught with difficulties. Not only is the exact theory of strong interactions intractable for neutron star conditions, but also, the properties of matter beyond nuclear density cannot be tested by direct experiment. Thus, neutron star observations are important in determining the behaviour of matter at high densities.

The mass of J1614-2230 provides a useful constraint on neutron star equations of state. Purely nucleonic equations of state are generally stiff enough not to be excluded. However, exotic equations of state are softer, and many long-standing exotic equations of state were excluded. This exclusion, along with theoretical motivations from particle physics, is guiding efforts to develop new exotic equations of state.

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